**Eigenfunctions/values of L (algebraically)**

So in the last lecture we determined the angular momentum eigenvalues and eigenvectors by projecting the eigenvalue/vector equation onto the position basis and solving the resulting PDE. The eigenvalues and eigenvectors of the operator, however, are the same no matter what basis we project the equation onto. So we shouldn’t *need* to use any projection. In fact the commutation relations among the Li’s are enough by themselves (three components and three equations) so that all that would be needed to add is a specification of the dimension of the matrices, which will come out later, indirectly. So let’s see how we can get the simultaneous eigenvectors and eigenvalues of and z without using any projection. When we do, we’ll encounter a surprising result.

**Determining the eigenvectors, Eigenvalues of L2, Lz via commutation relations**

So let’s call our simultaneous eigenvectors: or for short, such that:



Now we haven’t assumed the solution before we obtained it. Even though from the previous lecture we know that m is equal to an integer and ℓ is equal to a whole number, we’re not assuming that here. As far as we know, m can be any real number and ℓ can be any real number which makes the eigenvalues completely variable at this point.

OK let’s work on determining the eigenvectors/eigenvalues of the z operator first. As we begin, let’s recall how we did this with the Harmonic oscillator. There we had the equation:



we attempted to factor the Hamiltonian into two pieces H ~ a†a. The usefulness of these creation/annihilation operators was that applying the creation operator a† to |ψ­n> raised the energy of that state by ћω, and applying the annihilation operator, a, to |ψn> lowered the energy of that state by ћω. And in this way we built up the energy eigenstates and eigenvalues. If we look back over the derivation, the reason a and a† had those properties was because they satisfied a peculiar commutation relation with H, namely that:



which is almost like an eigenvalue equation if you step back from the equation a bit.

We would like to apply the same technique to our present eigenvalue problem. We want to identify our angular momentum creation/annihilation operators and use them to build up the excited states. Let’s start with z first. We want some linear combination of the angular momentum operators (since these are the only operators in the problem) so that our creation/annihilation operators are given by:



respectively. (going to leave off hats because jeez) + is our creation operator, ala a†, and - will be our annihilation operator, ala, a. Note that since we want -† = + in analogy with a and a†, the coefficients of - must be the complex conjugate of those of +. Alright, now since we’re trying to find the eigenstates and energies of z (akin to H in the analogous harmonic oscillator problem), we want these operators to satisfy:



where α is some number akin to the role of ћω in the Harmonic oscillator problem. Alright, so let’s plug in our ansatz for what L+ is.



So we see that we must have:



which implies that:



So our creation/annihilation operators (also called raising/lowering operators) are:



and if we work out the commutation relation between z and - we will see that all total, they obey the commutation relation, by design:



So it appears that our z-component momentum can be raised/lowered in increments of ћ. It follows from the same arguments as those used with the HO that the



and we’ll prove it again. For consider that:



So indeed +|ℓm> is an eigenfunction of z with angular momentum (m+1)ћ, which is the angular momentum that the state |ℓ,m+1> would have. So +|ℓm> must be proportional to |ℓ,m+1>. And similarly for -|ℓm>. You can show that it is an eigenfunction of z with angular momentum (m-1)ћ. So using the raising/lowering operators we can build up excited or ‘depressed?’ z eigenstates. Now let’s figure out what the raising/lowering operator does with the 2 eigenvalue. So let’s compute,



So since ± commutes with (it does because all of the components of commute with ), it doesn’t change the eigenvalue of the total angular momentum. Let’s work out the proportionality constant now. So say the constant of proportionality is c so that:



Then take the dot product of this vector with itself and we can work it out. We have:



Now we need to find another expression for what the product of these two operators is in order to explicitly evaluate their action (including the constant of proportionality) on the kets. So consider that:



So we have an important identity:



Now use this in our normalization expression,



and so we have explicitly that:



So this is major progress, but we still don’t know the allowed values of ℓ and m, though we do know what m can increase by (±1). But think about this. Suppose we keep applying the raising operator to an eigenstate. Then this will keep increasing the value of m, which will keep decreasing the value of the pre-factor, c, we just calculated. Now observe that the maximum value that m can take on must be ℓ, at which point c would be 0. If m could be any larger, then c would be imaginary which would make the normalization of the state <ℓ,m+1|ℓ,m+1> negative for instance – and this cannot be allowed. On the other hand, suppose that we keep applying the lowering operator to this state. This will decrease the value of m, which will keep decreasing the value of the prefactors. The minimum value that m can take on is the value of m such that the prefactor is zero. So this would be m = -ℓ.

Now by applying the raising operator to the state |ℓ,-ℓ>, the appropriate number of times, we must be able to get to the state |ℓ,ℓ>. Now we would have to apply the raising operator a total of 2ℓ times to go from the bottom state |ℓ,-ℓ> to the top |ℓ,ℓ>. So we must have that 2ℓ is a natural number: . Or in other words,



So we have that the general angular momentum eigenstates:



**Orbital angular momentum eigenstates in position space**

Focusing on the natural number values of ℓ for now, let’s project these eigenstates onto position space and see that they are the same as those we derived before. In practice, given a particular value of ℓ, we would start at the bottom of the rung – so to speak, with ψℓ,-ℓ(**r**). Since it is the lowest state, we must have:



Projecting this onto position space we’d have:



If we project the x and y angular momentum operators onto position space, like we did for the z and operators, then we’d get:



and consequently we’d have:



And so our equation would read,



I suppose we ought to try separation of variables to solve such an equation. So let ψ(r) = Θ(θ)Φ(φ). Then we have:



Both sides must be equal to a constant so:



This yields the solutions,



So all total we have:



now we know that m is the azimuthal quantum number so that m = -ℓ. So this translates to:





To perform the integral we can IBP (just making this up as I go). Let us define the integral as I(ℓ). Then we have:



So we have the recursion relation which can be solved:



And since



we get:



and therefore, solving for Aℓ we find the normalized wavefunction to be:



So for instance we would have:



which is indeed the formulae. If we want ψ10(**r**) for instance, then we can apply the raising operator to it.



and so,



which is also what we found before. Eigenstates further up the ladder can also be obtained by continuing to apply the raising operator L+. Conversely we can go the other way, using the lowering operator. We would have:



where we quickly write down the solution by comparing to the lowering operator equation with φ → -φ. Normalizing we get:



The (-1)ℓ doesn’t come directly from the normalization, which only fixes the eigenfunction up to an arbitrary phase [and (-1)ℓ is a phase]. The (-1)ℓ comes from the fact that it is required in order for 2ℓ applications of the raising operator on ψℓ,-ℓ(**r**) to match up to ψℓℓ(**r**). But generally we’ll just remember that:

